

LOWER BOUNDS ON THE BERGMAN METRIC NEAR POINTS OF INFINITE TYPE

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ABSTRACT. Let Ω be a pseudoconvex domain in \mathbb{C}^n satisfying an f -property for some function f , we show that the Bergman metric associated to Ω has the lower bound $\tilde{g}(\delta_\Omega(z)^{-1})$ where $\delta_\Omega(z)$ is the distance from z to $\partial\Omega$ and \tilde{g} is a specific function defined by f . This refines Khanh-Zampieri's work in [KZ12] with reducing the smoothness assumption of the boundary.

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1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{C}^n with the boundary $\partial\Omega$ and $\delta_\Omega(z)$ denote the distance from z to $\partial\Omega$. The Bergman metric associated to Ω at the point $z \in \Omega$ acting the vector $X \in T^{1,0}(\mathbb{C}^n)$ is defined by

$$B_\Omega(z, X) := \left(\sum_{j,k=1}^n \frac{\partial^2 \log K_\Omega(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \right)^{1/2}.$$

It is an interesting question is to consider how rate of the Bergman metric tends to infinity uniformly at the boundary points. When $\partial\Omega$ is C^2 -smooth and Ω is either strongly pseudoconvex or pseudoconvex of finite type in \mathbb{C}^2 , the Bergman metric $B_\Omega(z, X)$ is asymptotically equivalent to $\delta_\Omega^{-1/m}(z)|X^\tau| + \delta_\Omega^{-1}(z)|X^\nu|$ (see [Cat89, Mc92, Die70]) where X^τ and X^ν are the tangential and normal components of X and m is the type of the boundary ($m = 2$ if Ω is strongly pseudoconvex). For a general pseudoconvex domain in \mathbb{C}^n with C^∞ -smooth boundary, using the subelliptic estimate for the $\bar{\partial}$ -Neumann problem, McNeal [Mc92] gave a lower bound of this metric with rate $\delta_\Omega^{-\epsilon}(z)$ for some $\epsilon > 0$. This result was also obtained by Chen [Che02] by using the properties of plurisubharmonic peak functions in Hölder space. Recently, Herbort [Her14] proved that if an $(t^\epsilon\tilde{P})$ -property (see below) holds for Ω then $B_\Omega(z, X)$ has the lower bound $\delta_\Omega^{-\epsilon}(z)|\log(\delta_\Omega(z))|^{-M}|X|$ for some M . The novelty of the proofs by Chen and Herbort is that no smoothness assumptions of the boundary are made.

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It should be noted that by an amalgamation of results in [Cat83, Cat87, KZ10, KZ12, Kha14], if the boundary is smooth then the finite type condition, the subelliptic estimate for the $\bar{\partial}$ -Neumann problem, the existence of a family of plurisubharmonic peak functions in a Hölder space, the $(t^\epsilon\text{-}\tilde{P})$ -property, and the $(t^\epsilon\text{-}P)$ -property (see below) are equivalent.

For a general pseudoconvex domain Ω in \mathbb{C}^n that is not necessary of finite type nevertheless smooth boundary, Khanh-Zampieri [KZ10, KZ12] proved that if an $(f\text{-}P)$ -property with $\frac{f(t)}{\ln t} \rightarrow +\infty$ holds for Ω then the Bergman metric has a lower bound with the rate $\frac{f}{\log}(\delta_\Omega^{-1+\eta}(z))$ for any $\eta > 0$. The aim of this paper is to improve this result by reducing the assumption of smoothness of the boundary. Here is the main result of this paper.

Theorem 1.1. *Let Ω be a pseudoconvex domain in \mathbb{C}^n with C^2 -smooth boundary $b\Omega$ and z_0 be a boundary point. Assume that Ω has an $(f\text{-}P)$ -property at z_0 with f satisfying $\int_t^\infty \frac{da}{af(a)} < \infty$ for some $t \geq 1$ and denote the integral by $(g(t))^{-1}$. Then there exist a neighborhood U of z_0 and a constant $C > 0$ such that*

$$B_\Omega(z, X) \geq C \cdot \tilde{g}(\delta_\Omega^{-1}(z)) |X| \quad (1.1)$$

for any $z \in V \cap \Omega$ and $X \in T_z^{1,0}\mathbb{C}^n$, where the function \tilde{g} is given by

$$\tilde{g}(t) = \sqrt[4]{g(t^\gamma)} \text{ for all } t \geq 1$$

for some constant $\gamma > 0$.

The use of plurisubharmonic peak functions also enables to weaken the smoothness assumption on the boundary.

In what follows, \lesssim and \gtrsim denote inequalities up to a positive constant. Moreover, we will use \approx for the combination of \lesssim and \gtrsim . In addition, the superscript $*$ denotes the inverse function.

2. THE $(f\text{-}P)$ -PROPERTY AND PLURISUBHARMONIC PEAK FUNCTIONS

We start this section by the definition of the $(f\text{-}P)$ -property.

Definition 2.1. For a smooth, monotonic, increasing function $f : [1, +\infty) \rightarrow [1, +\infty)$ with $f(t)t^{-1/2}$ decreasing, we say that Ω has the $(f\text{-}P)$ -property (or f -property for short) if there exist a neighborhood U of $b\Omega$ and a family of functions $\{\phi_\delta\}$ such that

- (i) the functions ϕ_δ are plurisubharmonic, C^2 on U , and satisfy $-1 \leq \phi_\delta \leq 0$, and
- (ii) $i\partial\bar{\partial}\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2 -defining function of Ω .

If the bounded condition of ϕ_δ in (i) is replaced by the self-bounded gradient condition, i.e, $i\partial\bar{\partial}\phi_\delta \gtrsim i\partial\phi_\delta \otimes \bar{\partial}\phi_\delta$, we say that Ω has the $(f\text{-}\tilde{P})$ -property.

It is proven in [Cat87, Mc92] that if $\Omega \subset \mathbb{C}^n$ is of finite type, then Ω satisfies the t^ϵ -property. Therefore, the estimate (1.1) holds for $\tilde{g}(t) = t^\delta$ for some $\delta > 0$. Moreover, the f -property holds for a large class of infinite type pseudoconvex domains in \mathbb{C}^n , such as the following example:

Let $\Omega \subset \mathbb{C}^n$ be a domain defined by

$$\Omega = \left\{ z \in \mathbb{C}^n : \operatorname{Re}(z_n) + \sum_{j=1}^{n-1} P_j(z_j) < 0 \right\}, \quad (2.1)$$

where $\Delta P_j(z_j) \gtrsim \frac{\exp(-1/|x_j|^\alpha)}{x_j^2}$ or $\frac{\exp(-1/|y_j|^\alpha)}{y_j^2}$ for $j = 1, \dots, n-1$. Then the f -property holds with $f(t) = \log^{1/\alpha} t$ (see [Kha13]). As in [Kha13], we obtain the following corollary.

Corollary 2.2. a) *Let Ω be a pseudoconvex domain of finite type in \mathbb{C}^n . Then (1.1) holds for $\tilde{g}(t) = t^\epsilon$.*

b) *Let Ω be defined by (2.1) with $\alpha < 1$. Then (1.1) holds for $\tilde{g}(t) = \log^{(\frac{1}{\alpha}-1)/4} t$.*

The proof of Theorem 1.1 is based on the following result about the existence of a family of plurisubharmonic peak functions which was recently proven by Khanh [Kha13].

Theorem 2.3. *Under the assumptions of Theorem 1.1, for any $\zeta \in b\Omega$, there exists a C^2 plurisubharmonic function ψ_ζ on Ω which is continuous on $\overline{\Omega}$ and peaks at ζ (that means, $\psi_\zeta(z) < 0$ for all $z \in \overline{\Omega} \setminus \{\zeta\}$ and $\psi_\zeta(\zeta) = 0$). Moreover, there are some positive constants c_1 and c_2 such that the following hold for any constant $0 < \eta < 1$:*

- (1) $|\psi_\zeta(z) - \psi_\zeta(z')| \leq c_1 |z - z'|^\eta$ for any $z, z' \in \overline{\Omega}$; and
- (2) $g((-\psi_\zeta(z))^{-1/\eta}) \leq c_2 |z - \zeta|^{-1}$ for any $z \in \overline{\Omega} \setminus \{\zeta\}$.

The function ψ_ζ above is called a plurisubharmonic peak function at the boundary point ζ . The following lemma follows immediately from Theorem 2.3.

Corollary 2.4. *Under the assumptions of Theorem 2.3, for any $\zeta \in b\Omega$ there are some positive constants c_1 and c_1 such that the following hold for any constant $0 < \eta < 1$:*

$$-c_1 |z - \zeta|^\eta \leq \psi_\zeta(z) \leq - \left(\frac{1}{g^*(c_2/|z - \zeta|)} \right)^\eta,$$

where ψ_ζ is the plurisubharmonic peak function given in Theorem 2.3.

We also need a version of L^2 -estimate for the $\bar{\partial}$ -equation that is generalized by Berndtsson, due to Donnelly - Fefferman [DF83]

Proposition 2.5. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n and let φ be plurisubharmonic in Ω . Let ψ be plurisubharmonic and assume that ψ has a self-bounded gradient. Let $0 < \nu < 1$. Then for any $\bar{\partial}$ -closed $(0,1)$ -form g in Ω , there is a solution u to the equation $\bar{\partial}u = g$ such that*

$$\int_{\Omega} |u|^2 e^{-\varphi + \nu\psi} dV \leq \frac{4}{\nu(1-\nu)^2} \int_{\Omega} |g|_{\bar{\partial}\bar{\partial}\psi}^2 e^{-\varphi + \nu\psi} dV.$$

3. BOUNDARY BEHAVIOR OF THE BERGMAN METRIC

Combining with the result in Theorem 2.3, we can now rephrase Theorem 1.1 in a more general setting. More precisely, we have the following theorem, which generalizes [Che02, Theorem 2].

Theorem 3.1. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n , ζ be a given boundary point, F_1 and F_2 are positive increasing functions satisfying that the function $-\log \circ F_1^*(-t)$ is convex on $(-1, 0)$. Assume that on a neighborhood U of ζ , there is a plurisubharmonic function ρ_ζ peaking at ζ and satisfying*

$$-F_1(|z - \zeta|) \leq \rho_\zeta(z) \leq -F_2(|z - \zeta|) \tag{3.1}$$

for $z \in U \cap \Omega$. Then there exists $k_0 > 0$ such that

$$B_\Omega(z, X) \gtrsim (F_2^*(F_1((3\delta(z))^{1/k_0})))^{-1/4} |X|.$$

Furthermore, in the case where the plurisubharmonic peak function satisfies a lower bound, we obtain the following theorem, which is a generalization of [Che02, Theorem 1].

Theorem 3.2. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n , w_0 be a given boundary point, F is positive increasing function satisfying that the function $-\log \circ F^*(-t)$ is convex on $(-1, 0)$. Assume that on a neighborhood U of w_0 , there is a plurisubharmonic function ρ peaking at w_0 and satisfying*

$$-F(|z - w_0|) \leq \rho(w) \quad (3.2)$$

for $z \in U \cap \Omega$. Then

$$\inf_{0 \neq X \in T^{1,0}\mathbb{C}^n} B_\Omega(z, X)/|X| \rightarrow +\infty$$

as $z \rightarrow w_0$.

The proofs of Theorem 3.1 and Theorem 3.2 are adapted from the argument by Chen [Che02] with precise the rate of lower bounds.

Let z be a fixed point in Ω , and $\pi(z)$ be the projection of z to the boundary $b\Omega$. Put $\phi(z) := -\log(F^*(-\rho(z)))$. Notice that since $-\log \circ F^*(-t)$ is convex on and increasing on $(-1, 0)$, we have that ϕ is plurisubharmonic on U . Moreover, by shrinking the neighborhood U , we can assume that $F_1(|z - w|) < 1$ and $F(|z - w|) < 1$ for any $z \in U \cap \Omega$.

Denote by $\delta(z) := \delta_\Omega(z) = |z - \pi(z)|$ the Eucildian distance from z to $b\Omega$. For $k \in \mathbb{N}^*$, we define on Ω a function as follows

$$g_{k,w}(z) = \chi \left(\frac{1}{\log k} (-\log \phi(z) + \log(-\log \epsilon) + 1) \right) \log |z - w|,$$

where χ is a $C^\infty(\mathbb{R})$ cut-off function satisfying

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \end{cases}$$

such that $\sup |\chi'| \leq 2$ and $\sup |\chi''| \leq 1$.

To prove the Theorem 3.1 and Theorem 3.2, we need the following lemma.

Lemma 3.3. *Let $C > 2n$ be a positive constant. Then there exists a constant $k_0 > 0$ depends on F and n so that for any $w \in \Omega$ with $|w - w_0| < \epsilon^{k_0}/2$ the following holds*

- i) $g_{k_0,w}(w) \approx \log |z - w|$ when z near w ;
- ii) $4Cg_{k_0,w}(w) + \phi(z) - \log(-\log |w - z|)$ is a plurisubharmonic function on Ω .

Proof. We may assume that $|w - w_0| < \epsilon^k/2$ where k will be determined later on. Then we have

$$\rho(w) \geq -F(|w - w_0|) \geq -F(\epsilon^k/2) > -F(\epsilon^k).$$

From the definition of the cut-off function χ and $g_{k,w}$, we get

$$\{\zeta \in \Omega : \rho(\zeta) > -F(\epsilon^k)\} \subset \{z \in \Omega : g_{k,w}(z) = \log |z - w|\}.$$

Thus we conclude that $g_{k,w}(z) \sim \log |z - w|$ near w .

A computation shows that

$$\partial\bar{\partial}g_{k,w} = \frac{\log|z-w|}{\phi \log k} \left(\chi''(\cdot) \frac{\partial\phi\bar{\partial}\phi}{\phi \log k} + \chi'(\cdot) \frac{\partial\phi\bar{\partial}\phi}{\phi} - \chi'(\cdot) \partial\bar{\partial}\phi \right) \quad (3.3)$$

$$- \frac{\chi(\cdot) \log|z-w|}{\phi \log k} \left(\partial\phi \frac{\bar{\partial}\log|z-w|}{\log|z-w|} + \bar{\partial}\phi \frac{\log|z-w|}{\log|z-w|} \right) \quad (3.4)$$

$$+ \chi(\cdot) \partial\bar{\partial}\log|z-w|. \quad (3.5)$$

It is clearly that the term (3.5) is non-negative and thus it can be neglected. For another terms, it is sufficient to consider them in the support of χ' . Moreover, we have

$$\text{supp}\chi' \subset \{z \in \Omega : \rho(z) < -F(\epsilon^k)\} \subset \{z \in \Omega : |z - w_0| \geq \epsilon^k\}$$

Therefore, one obtains

$$|z - w| \geq |z - w_0| - |w - w_0| \geq \frac{1}{2}|z - w_0|$$

on $\text{supp}\chi'(\cdot)$ since $|w - w_0| < \epsilon^k/2$, and hence

$$|\phi(z)| = |\log F^*(-\rho(z))| \geq |\log|z - w_0|| \geq |\log 2||z - w|.$$

Now Cauchy-Schwarz's inequality implies that

$$\pm 2\text{Re} \left(\frac{\partial\phi\bar{\partial}\log|z-w|}{\log|z-w|} \right) \geq -\partial\phi\bar{\partial}\phi - \partial\log(-\log|z-w|)\bar{\partial}\log(-\log|z-w|)$$

in the distribution sense.

Combining above statements, there exists a constant C' (depending only on F) so that

$$\partial\bar{\partial}g_{k,w}(z) \geq -\frac{C'}{\log k} (\partial\bar{\partial}\phi + \partial\bar{\partial}(-\log(-\log|z-w|)))$$

provided $\phi > \log 2$ on Ω , $\partial\bar{\partial}\phi \geq \partial\phi\bar{\partial}\phi$, and

$$\partial\bar{\partial}(-\log(-\log|z-w|)) \geq \partial\log(-\log|z-w|)\bar{\partial}\log(-\log|z-w|).$$

Therefore, if we take k_0 big enough so that $\frac{C'}{\log k_0} < \frac{1}{4C}$, then the assertion (ii) follows. \square

Proof of Theorem 3.2. We shall follow the guildlines of [Che02]. First of all, we recall that

$$B_\Omega(w, X) = K^{-1/2}(w) \sup\{|Xf(w)| : f \in H^2(\Omega), f(w) = 0 \text{ and } \|f\|_\Omega \leq 1\}.$$

Let $X \in T^{1,0}(\mathbb{C}^n)$. Since the Bergman metric is biholomorphically invariant, without loss of generality we may assume that $X = |X|\partial_{w_1}$.

Recall that $\phi = -\log(F^*(-\rho))$ and define

$$\delta = \delta(\epsilon) := \sup_{z \in \bar{\Omega}, \rho(z) \geq -F(\epsilon)} |z - w_0|.$$

Note that $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$ since ρ is a plurisubharmonic peak function. Furthermore, without loss of generality we can assume that $U \cap \Omega = \Omega$, $\text{diam}(\Omega) \leq e^{-1}$, and $\phi > \log 2$.

We now define

$$\eta_w = \kappa \left(-\log(-\log |z - w|) + \log(-\log \delta^{1/2}) + 1 \right),$$

$$\psi_w = \frac{1}{2} (\phi - \log(-\log |z - w_0|)),$$

and

$$\varphi_w = Cg_w - \frac{1}{4} \log(-\log(|z - w|)) + \frac{1}{2} \psi_w,$$

where κ is a cut-off function such that

$$\kappa(x) = \begin{cases} 1 & \text{if } x < 1 - \log 2, \\ 0 & \text{if } x > 1; \end{cases}$$

and $C > 2n$ comes from Lemma 3.3. It is easy to check by a simple computation that φ_w is plurisubharmonic, $\partial \bar{\partial} \psi_w \gtrsim \partial \psi_w \bar{\partial} \psi_w$ and $\partial \bar{\partial} \psi_w \gtrsim \partial \log |z - w| \bar{\partial} \log |z - w|$. Then we obtain

$$|\bar{\partial} \eta_w|_{\partial \bar{\partial} \psi_w} \leq \sup |\kappa'|.$$

Noting that $\text{supp } g_w \subset \{z \in \Omega : \rho(z) \geq -F(\epsilon)\} \subset \{z \in \Omega : \eta_w(z) = 1\}$. This implies that $\eta_w(w) = 1$ and $\text{supp } \bar{\partial} \eta_w \cap \text{supp } g_w = \emptyset$.

Now, we apply Proposition 2.5 with $\nu = \frac{1}{2}$, $\varphi = \varphi_w$ and $\psi = \psi_w$ to solve the $\bar{\partial}$ -equation

$$\bar{\partial} u_w = (z_1 - w_1) \frac{K_\Omega(z, w)}{K_\Omega^{1/2}(w)} \bar{\partial} \eta_w,$$

on Ω with the estimate

$$\begin{aligned} & \int_{\Omega} |u_w|^2 e^{-Cg_w + \frac{1}{4} \log(-\log |z - w|)} dV \\ & \lesssim \int_{\text{supp } \bar{\partial} \eta_w} |z_1 - w_1|^2 \frac{|K_\Omega^2(z, w)|}{|K_\Omega(w)|} |\bar{\partial} \eta_w|_{\partial \bar{\partial} \psi_w}^2 e^{-Cg_w + \frac{1}{4} \log(-\log |z - w|)} dV \\ & \leq C_1 \int_{\Omega \cap \{|z - w_0| < \delta^{1/2}\}} |z - w|^2 \frac{|K_\Omega(\zeta, z)|^2}{|K_\Omega(z)|} dV \\ & \leq C_2 \delta^{1/2}, \end{aligned}$$

where C_1 and C_2 are constants only depending on $\sup \kappa'$.

Since $Cg_w - \frac{1}{4} \log(-\log |z - w|) < 0$ on Ω , $g_w \sim \log |z - w|$ near w , and

$$Cg_w - \frac{1}{4} \log(-\log |z - w|) < 2n \log |z - w|$$

near w , it follows that $u_w(w) = 0$ and the function

$$f_w = (z_1 - w_1) \frac{K_\Omega(z, w)}{K_\Omega^{1/2}(w)} \eta_w - u_w$$

is holomorphic on Ω and satisfies

$$f_w(w) = u_w(w) = 0 \text{ and } X f_w(z) = |X| K_\Omega^{1/2}(w);$$

$$\begin{aligned}
\|f_w\|_\Omega &\leq \left\| (z_1 - w_1) \frac{K_\Omega(\cdot, w)}{K_\Omega^{1/2}(w)} \eta_w \right\|_\Omega + \|u_w\|_\Omega \\
&\leq C_3 \delta^{1/2} + \|u_z\|_{1/2 C g_w - 1/2 \log(-\log|z-w|)} \\
&\leq C_4 \delta^{1/4}.
\end{aligned}$$

Define $h_w = \frac{f_w}{\|f_w\|}$. Then we have h_w is also holomorphic, $h_w(w) = 0$, and $\|h_w\| = 1$. Therefore, we conclude

$$B_\Omega(w, X) \geq \frac{|X h_w(w)|}{K_\Omega^{1/2}(w)} = \frac{|X h_w(w)|}{K_\Omega^{1/2}(w) \|f_w\|_\Omega} \geq C_4^{-1} \delta^{-1/4} |X|$$

for any $w \in \Omega \cap \{w \in \Omega: |w - w_0| < \epsilon^{k_0}/2\}$. So, the proof is complete. \square

Proof of Theorem 3.1. We shall repeat the argument as in the proof of Theorem 3.2. For any $w \in \Omega$, let $w' := \pi(w)$. It means that $|w - w'| = \delta_\Omega(w)$. Then we take $\epsilon := (3\delta_\Omega(w))^{1/k_0}$. Hence, it is clear that $|w - w'| < \epsilon^{k_0}/2$ and

$$\delta = \sup_{z \in \overline{\Omega}, \rho_{w'}(z) \geq -F_1(\epsilon)} |z - w'| \leq F_2^*(F(\epsilon))$$

because $-\rho_{w'}(z) \geq F_2(|z - w'|)$. Therefore, we obtain

$$B_\Omega(w, X) \gtrsim \delta_\Omega^{-1/4}(w) |X| \gtrsim (F_2^*(F_1((3\delta_\Omega(w))^{1/k_0})))^{-1/4} |X|,$$

which proves the theorem. \square

We now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote $F_1(t) := c_1 t^\eta$ and $F_2(t) := \left(\frac{1}{g^*(c_2/t)}\right)^\eta$ for all $t \geq 1$, where $0 < \eta < 1$. Then, a computation shows that

$$F_2^*(F_1((3\delta(z))^{1/k_0})) = c_2/g \left(\frac{1}{c_1^{1/\eta} (3\delta(z))^{1/k_0}} \right).$$

Therefore, by Corollary 2.4 and employing Theorem 3.1 for $F_1(t) = c_1 t^\eta$ and $F_2(t) = \left(\frac{1}{g^*(c_2/t)}\right)^\eta$, where $\eta \in (0, 1)$ is given in Corollary 2.4, we obtain

$$B_\Omega(z, X) \gtrsim \left(g \left(\frac{1}{c_1^{1/\eta} (3\delta(z))^{1/k_0}} \right) \right)^{1/4} |X|$$

for any $z \in V \cap \Omega$ and $X \in T_z^{1,0} \mathbb{C}^n$. Moreover, by the increasing of g and decreasing of $g(t)/t$, we conclude that

$$B_\Omega(z, X) \gtrsim \tilde{g}(\delta_\Omega(z)) |X|$$

for any $z \in V \cap \Omega$ and $X \in T_z^{1,0} \mathbb{C}^n$, where the function \tilde{g} is given by

$$\tilde{g}(t) = \sqrt[4]{g(t^{1/k_0})} \text{ for every } t \geq 1.$$

Hence, the proof is complete. \square

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